

A note on sub-total domination in graphs

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Abstract

Let G be a simple and finite graph without isolated vertices. In this note we study a degree sequence derived invariant called the *sub-total domination number*, denoted $\text{sub}_t(G)$. This invariant originally appeared in [10] and serves as a lower bound on $\gamma_t(G)$, where $\gamma_t(G)$ denotes the heavily studied *total domination number* of G .

Keywords: Total dominating sets; total domination number; sub-total domination number, degree sequence index strategy

AMS subject classification: 05C69

1 Introduction

Domination in graphs is widely studied and a heavily applied notion in graph theory. Indeed, domination and its variants and generalizations appear in vast quantities in the mathematical literature; see for example [3, 6, 8, 12, 13, 14, 16, 24, 26]. Of the many variants of domination, total domination is arguably one of the most natural. Given a graph G , and a set of vertices S in G , S is a *total dominating set* if every vertex in G has a neighbor in S . The minimum cardinality of a total dominating set in G is the *total domination number* of G , denoted by $\gamma_t(G)$. It is well known that determining the total domination number of a general graph is in the class of NP -complete decision problems [25], and as such, a significant amount of research has been devoted to finding easily computable upper and lower bounds on $\gamma_t(G)$; see for example the monograph [16] which details and surveys total domination.

As previously mentioned, finding computationally efficient bounds on $\gamma_t(G)$ is desired. However, in a much more general fashion, it is of great interest to find computationally efficient bounds for any *NP*-hard graph invariant. With this in mind, we make note that the degree sequence of a graph has been shown to yield such desired bounds. Two well known examples are the *residue* and the *annihilation number* of a graph, which serve as respective lower and upper bounds on the computationally difficult *independence number* of a graph [7, 23]. With regards to domination, the lesser known degree sequence derived invariants known as the *slater number* and the *sub- k -domination number* serve as respective lower bounds on the *domination number* and *k -domination number* of a graph [1, 27]. We remark that these degree sequence results are special cases of the recently introduced *degree sequence index strategy* (DSI-strategy) [2].

Definitions and Notation. All graphs in this paper will be considered finite simple graphs without isolated vertices. Let $G = (V, E)$ be a graph. We will denote the order and size of G by $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$, respectively. When the dependence on G is clear, we will write n in place of $n(G)$. Two vertices $v, w \in V(G)$ are said to be neighbors if $vw \in E(G)$. The open neighborhood of $v \in V(G)$, denoted by $N_G(v)$, is the set of neighbors of v , whereas the close neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v \in V(G)$ is the cardinality of $N_G(v)$, and will be denoted by $d_G(v)$. The maximum and minimum vertex degree among all vertices of G will be denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is called k -regular if $d_G(v) = k$ for all $v \in V(G)$. A regular graph is a graph that is k -regular for some integer $k \geq 0$.

The degree sequence of G , is the sequence consisting of the vertex degrees in G listed in non-increasing order, and will be denoted $D(G) = \{\Delta(G) = d_1, \dots, d_n = \delta(G)\}$. For brevity, we may write the number of vertices realizing each degree in superscript. For example, the path P_n , on n vertices, may have degree sequence written $D(P_n) = \{2^{n-2}, 1^2\}$. If a sequence of non-negative integers D has the property that $D = D(G)$, for some graph G , then we say that D is a graphic sequence, and that D is realizable by G . We note that a given graphic sequence may have more than one graph which realizes D .

A set of vertices $S \subseteq V(G)$ is a total dominating if every vertex in G has a neighbor in S , and such a set will be called a TD-set of G . The cardinality of a smallest TD-set in G is the total domination number of G , denoted by $\gamma_t(G)$, and such a set will be called a $\gamma_t(G)$ -set. For other graph terminology and definitions, we will follow [16].

We will also make use of the notation $[k] = \{1, \dots, k\}$.

2 Sub-total domination

In this section we present our main results. First we recall the definition of the sub-total domination number, originally defined in [10], and denoted $sl_t(G)$. Keeping our

notation and terminology consistent with [1], we will use $\text{sub}_t(G)$ in place of $sl_t(G)$.

Definition 1 *If G is an isolate-free graph with order n and degree sequence $D(G) = \{\Delta(G) = d_1, \dots, d_n = \delta(G)\}$, the sub-total domination number $\text{sub}_t(G)$, is defined as the smallest integer k such that $\sum_{i=1}^k d_i \geq n$.*

With the definition of sub-total domination now defined, we remark that $\text{sub}_t(G)$ can be computed in $O(n)$ time. Because of the simplicity of computing $\text{sub}_t(G)$, and the difficulty of computing $\gamma_t(G)$, the following theorem serves as one of our main results. We remark that this theorem first appeared in [10] without proof.

Theorem 1 ([10]) *If G is an isolate-free graph, then*

$$\gamma_t(G) \geq \text{sub}_t(G),$$

and this bound is sharp.

Proof. Let G be a graph with order n , degree sequence $D(G) = \{\Delta(G) = d_1, \dots, d_n = \delta(G)\}$, and S be a $\gamma_t(G)$ -set. Next, we order the vertices of S , $s_1, \dots, s_{|S|}$, so that $d_G(s_1) \geq \dots \geq d_G(s_{|S|})$. By definition, every vertex is totally dominated by a vertex in S ; that is, every vertex has a neighbor in S . Thus, $V(G) = \cup_{v \in S} N_G(v)$, which implies,

$$n = \left| \bigcup_{v \in S} N_G(v) \right| \leq \sum_{v \in S} |N_G(v)| = \sum_{i=1}^{|S|} d_G(s_i).$$

In particular, we have established,

$$\sum_{i=1}^{|S|} d_G(s_i) \geq n.$$

Next observe that the i -th term of $D(G)$ is greater than or equal to the i -th degree of the list of vertices from S , and thus, we have the following inequality,

$$\sum_{i=1}^{|S|} d_i \geq \sum_{i=1}^{|S|} d_G(s_i) \geq n.$$

That is,

$$\sum_{i=1}^{|S|} d_i \geq n. \tag{1}$$

Since $\text{sub}_t(G)$ is the smallest integer satisfying (1), it follows that $\gamma_t(G) = |S| \geq \text{sub}_t(G)$, and the lower bound has been proven.

To see that this bound is sharp, consider the star $K_{1,n-1}$ on $n \geq 2$ vertices. Then, $\gamma_t(K_{1,n-1}) = 2$, and $\text{sub}_t(K_{1,n-1}) = 2$. \square

Theorem 1 is sharp for non-trivial stars. However, stars are a special case of a more general concept. Namely, if G is a connected graph with order $n \geq 2$ and maximum degree $\Delta(G) = n - 1$, then choosing a maximum degree vertex and an arbitrary neighbor of this vertex forms a TD-set, and hence, $\gamma_t(G) = 2$. Moreover, the highest vertex degree summed with the next highest vertex degree will be greater than n , and so $\text{sub}_t(G) = 2$. In particular, since no vertex of G will have degree n , it follows that $\text{sub}_t(G) \geq 2$. We combine these ideas with the following proposition.

Proposition 2 *If G is a connected graph with order $n \geq 2$ and maximum degree $\Delta(G) = n - 1$, then $\gamma_t(G) = \text{sub}_t(G) = 2$.*

There exists graphs G for which $\gamma_t(G) = 2$ and $\Delta(G) \neq \Delta(G) - 1$. Double stars (trees with exactly two non-leaf vertices) are one such example. With this in mind, we next generalize Proposition 2 to a statement on graphs G with $\gamma_t(G) = 2$. That is, since $\text{sub}_t(G) \geq 2$, we obtain the following proposition.

Proposition 3 *If G is an isolate-free graph with $\gamma_t(G) = 2$, then $\gamma_t(G) = \text{sub}_t(G)$.*

A simple lower bound on the total domination number of isolate-free graphs can be found by dividing the order by the maximum degree, see Chapter 2, Theorem 2.11. in [16]. With the following theorem we show that the sub-total domination number improves on this bound.

Theorem 4 *If G is an isolate-free graph with order n and maximum degree $\Delta(G)$, then $\gamma_t(G) \geq \text{sub}_t(G) \geq n/\Delta(G)$.*

Proof. Let G be an isolate-free graph with order n and maximum degree $\Delta(G)$. The left hand side of the inequality is a restatement of Theorem 1. Thus, in order to prove this result, it suffices to show $\text{sub}_t(G) \geq n/\Delta(G)$. By definition, we have

$$\sum_{i=1}^{\text{sub}_t(G)} d_i \geq n.$$

Next observe that $\Delta(G) \geq d_i$ for each $i \in [\text{sub}_t(G)]$, and thus

$$\text{sub}_t(G)\Delta(G) = \sum_{i=1}^{\text{sub}_t(G)} \Delta(G) \geq \sum_{i=1}^{\text{sub}_t(G)} d_i \geq n,$$

Hence, $\text{sub}_t(G) \geq n/\Delta(G)$, and the proof of the theorem is complete. \square

3 Properties of $\text{sub}_t(G)$

In this section we provide various fundamental properties of the sub-total domination number. We begin with a closed formula for $\text{sub}_t(G)$ in the case that G is isolate-free and k -regular.

Proposition 5 *If $k \geq 1$ is an integer and G is a k -regular graph with order n , then $\text{sub}_t(G) = \lceil n/k \rceil$.*

Proof. Let $k \geq 1$ be an integer, and let G be a k -regular isolate-free graph with order n . By definition of sub-total domination, we have

$$\text{sub}_t(G)k = \sum_{i=1}^{\text{sub}_t(G)} k \geq n.$$

It follows that $\text{sub}_t(G) \geq n/k$. Since $\text{sub}_t(G)$ is the smallest integer satisfying this inequality, we obtain $\text{sub}_t(G) = \lceil n/k \rceil$, and the proof of the proposition is complete. \square

Next we consider sub-total domination of disjoint isolate-free graphs. In particular, we show that sub-total domination is subadditive with respect to disjoint unions of graphs.

Lemma 6 *If G and H are isolate-free graphs, then $\text{sub}_t(G) + \text{sub}_t(H) \geq \text{sub}_t(G \cup H)$.*

Proof. Let G and H be disjoint graphs with degree sequences $D(G) = \{\Delta(G) = d_1^G, \dots, d_{n_1}^G = \delta(G)\}$ and $D(H) = \{\Delta(H) = d_1^H, \dots, d_{n_2}^H = \delta(H)\}$. By definition of sub-total domination, we have

$$\sum_{i=1}^{\text{sub}_t(G)} d_i^G \geq n_1,$$

and,

$$\sum_{i=1}^{\text{sub}_t(H)} d_i^H \geq n_2.$$

Thus,

$$\sum_{i=1}^{\text{sub}_t(G)} d_i^G + \sum_{i=1}^{\text{sub}_t(H)} d_i^H \geq n_1 + n_2 = n(G \cup H).$$

Denote the degree sequence of $G \cup H$ by $D(G \cup H) = \{\Delta(G \cup H) = d_1^*, \dots, d_{n_1+n_2}^* = \delta(G \cup H)\}$. Since degree sequences are listed in non-increasing order, it follows that

$$\sum_{i=1}^{\text{sub}_t(G)+\text{sub}_t(H)} d_i^* \geq \sum_{i=1}^{\text{sub}_t(G)} d_i^G + \sum_{i=1}^{\text{sub}_t(H)} d_i^H \geq n_1 + n_2 = n(G \cup H).$$

That is,

$$\sum_{i=1}^{\text{sub}_t(G)+\text{sub}_t(H)} d_i^* \geq n(G \cup H). \quad (2)$$

Since $\text{sub}_t(G \cup H)$ is the smallest integer satisfying (2), it follows that $\text{sub}_t(G \cup H) \leq \text{sub}_t(G) + \text{sub}_t(H)$, and the proof of the lemma is complete. \square

It is easy to see that the total domination number is additive with respect to unions of disjoint graphs; that is, for disjoint isolate-free graphs G and H , $\gamma_t(G \cup H) = \gamma_t(G) + \gamma_t(H)$. With this in mind, the following theorem serves as an improvement on Theorem 1 when considering the union of disjoint graphs.

Theorem 7 *If G and H are isolate-free graphs, then*

$$\gamma_t(G \cup H) \geq \text{sub}_t(G) + \text{sub}_t(H) \geq \text{sub}_t(G \cup H).$$

Proof. Let G and H be isolate-free graphs. By Lemma 6 $\text{sub}_t(G) + \text{sub}_t(H) \geq \text{sub}_t(G \cup H)$. Moreover, since total domination is additive with respect to disjoint unions, $\gamma_t(G \cup H) = \gamma_t(G) + \gamma_t(H)$. By Theorem 1, $\gamma_t(G) \geq \text{sub}_t(G)$ and $\gamma_t(H) \geq \text{sub}_t(H)$. Thus, $\gamma_t(G \cup H) \geq \text{sub}_t(G) + \text{sub}_t(H)$, and the theorem is proven. \square

4 Conclusion and Open Problems

In this note we have studied fundamental properties of $\text{sub}_t(G)$. However, we have not studied many classes of graphs for which $\gamma_t(G) = \text{sub}_t(G)$. Since $\text{sub}_t(G)$ is easily computable, we suggest the following problem.

Problem 1 *Characterize all graphs G for which $\gamma_t(G) = \text{sub}_t(G)$.*

Problem 1 is surely difficult, and leads to the question of asking if determining a graph G satisfies $\gamma_t(G) = \text{sub}_t(G)$ is *NP*-complete. The analogous question for sub-domination and domination is known to be *NP*-complete [10], and so this provides evidence that this may indeed be the case.

There exists many lower bounds on the total domination number of a graph, and it remains to be shown how sub-total domination compares with most of these bounds. Thus, we further suggest the following problem.

Problem 2 *Compare $\text{sub}_t(G)$ with known lower bounds on $\gamma_t(G)$.*

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